2.3 Expansions of scattered waves

Resonances describe oscillations & decay of waves for problems on non-compact domains, replacing the usual eigenvalues & Fourier expansions.

{ Throughout V is real-valued, so spectral theory for selfadjoint operators can be used.

Compact domains: Consider the following eigenvalue problem on $[a,b]$.

$$\begin{cases} P_V u = \lambda^2 u & \text{on } (a,b) \\ u(a) = u(b) = 0 \end{cases} \qquad \text{where } P_V = D_x^2 + V$$

There is a discrete set of λ s.t. we can solve this. Specifically, the problem has a set of distinct sol'ns $(i\sqrt{-E_k}, v_k)$, (λ_j, u_j) where $E_N < \dots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \dots \longrightarrow \infty$ and the sol'ns u_j, v_k are L^2-normalized.

For an example of eigenfunction expansion ("generalized Fourier expansion" - legitimately Fourier series when $V \equiv 0$), consider the wave equ'n

$$\begin{cases} (D_t^2 - P_V) w = 0 & \text{on } \mathbb{R} \times (a,b) \\ w(0,x) = w_0(x),\ \partial_t w(0,x) = w_1(x) & \text{on } [a,b] \\ w(t,a) = w(t,b) = 0. \end{cases}$$

The sol'n w can be written as:

$$w(t,x) = \sum_{k=1}^{N} \cosh(t\sqrt{-E_k})\, a_k v_k(x) + \sum_{k=1}^{N} \frac{\sinh(t\sqrt{-E_k})}{\sqrt{-E_k}} b_k v_k(x)$$
$$+ \sum_{j=0}^{\infty} \cos(t\lambda_j)\, c_j u_j(x) + \sum_{j=0}^{\infty} \frac{\sin(t\lambda_j)}{\lambda_j} d_j u_j(x).$$

Where $a_k = \int_a^b w_0(x)\overline{v_k}(x)\,dx$ $\quad b_k = \int_a^b w_1(x)\overline{v_k(x)}\,dx$

$c_j = \int_a^b w_0(x)\overline{u_j}(x)\,dx$ $\quad d_j = \int_a^b w_1(x)\overline{u_j(x)}\,dx.$

and the rest of the data come from the stationary equ'n. [the equ'n Fourier transformed in t].

It would now be desirable to replace the compact set $[a,b]$ by a non-compact set. What better noncompact set than $\mathbb{R}$?

Thm 2.9 (Resonance expansions of scattering waves in one dimension)

Let $V \in L^\infty(\mathbb{R};\mathbb{R})$ and suppose that $w(t,x)$ is the sol'n of

$$(2.3.2)\quad \begin{cases} (D_t^2 - P_V)w(t,x) = 0, \\ w(0,x) = w_0(x) \in H^1_{comp}(\mathbb{R}) \\ \partial_t w(0,x) = w_1(x) \in L^2_{comp}(\mathbb{R}) \end{cases}$$

Then, for any $A>0$, $w(t,x) = \sum_{\mathrm{Im}\lambda_j > -A} \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) + E_A(t)$,

where the sum is finite,

$$(2.3.4)\quad \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) = -\mathrm{Res}_{\mu=\lambda_j}\left((iR_V(\mu)w_1 + \mu R_V(\mu)w_0)e^{-i\mu t} \right)$$

$$(P_V - \lambda_j^2)^{\ell+1} f_{j,\ell} = 0,$$

and for any $K>0$ such that $\operatorname{supp} w_j \subseteq (-K,K)$, $\exists$ constants $C_{K,A}$, $T_{K,A}$ s.t.

$$\|E_A(t)\|_{H^2([-K,K])} \le C_{K,A}\, e^{-tA}\left(\|w_0\|_{H^1} + \|w_1\|_{L^2}\right), \quad t \ge T_{K,A}.$$

error is controlled by initial data and supp for large times.

Recall $R_V(\mu) := (D_x^2 + V - \mu^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the resolvent operator Yonah spoke on.

The regularity of the error term can be explained by propagation of singularities — as $t \to \infty$ singularities will leave each fixed compact set. In particular, when $V \in C_c^\infty(\mathbb{R})$ the proof will show that we can replace $\|E_A(t)\|_{H^2([-k,k])}$ by $\|E_A(t)\|_{H^k([-k,k])}$ $\forall k$.

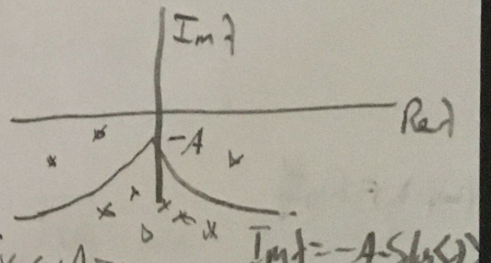
The proof of this theorem relies on another theorem which shows existence of a resonance free region and gives an estimate for the resolvent.

Theorem 2.10 (Resonance free regions in one-dimension):

Suppose that $V \in L^\infty_{comp}(\mathbb{R};\mathbb{C})$. Then $\forall \rho \in C_c^\infty(\mathbb{R})$ and $\delta < \frac{1}{|\mathrm{chsupp}\, V|}$ ($\mathrm{chsupp}\, V$ is the convex hull of the support of V) $\exists$ constants A, C, T s.t.

(2.3.5) $$\|\rho R_V(\lambda)\rho\|_{L^2 \to H_j} \le C|\lambda|^{j-1} e^{T(\mathrm{Im}\,\lambda)_-} \quad j=0,1,2,$$

for $\mathrm{Im}\,\lambda \ge -A - \delta \log(1+|\lambda|)$, $|\lambda| > C_0$



In particular there are only finitely many resonances in the region $\{\lambda : \mathrm{Im}\,\lambda \ge -A - \delta \log(1+|\lambda|)\}$ for any $A > 0$

Pf From Yonah's section (Theorem 2.1) we have the estimate

(2.3.6) $$\|\rho_1 R_0(\lambda)\rho_1\|_{L^2 \to L^2} \le C|\lambda|^{-1} e^{(b-a)|\mathrm{Im}\,\lambda|}, \quad \rho_1 \in L^\infty \ \mathrm{supp}\,\rho_1 \subseteq [a,b].$$

Also from the previous section we have the identity

(2.3.7) $$\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho_1 (I + VR_0(\lambda)\rho_1)^{-1} (I - VR_0(\lambda)(1-\rho_1))\rho$$

where $\rho \equiv 1$ on $\operatorname{supp} V$, $\rho_1 \in L^\infty_{comp}(\mathbb{R})$ satisfies $\rho_1 V = V$ [e.g. $\rho_1 = \mathbb{1}_{\operatorname{chsupp}(V)}$]. Now we want to invert $I + VR_0(\lambda)\rho_1$ by Neumann series, so must show that $\|V\rho_1 R_0(\lambda)\rho_1\|_{L^2 \to L^2} \leq 1/2$

To see this, put $[a,b] := \operatorname{chsupp} V$ and use (2.3.6) to see that

$$\|V\rho_1 R_0(\lambda)\rho_1\|_{L^2 \to L^2} \leq C\|V\|_{L^\infty} e^{(b-a)|\operatorname{Im}\lambda|}/|\lambda|$$

which if $\operatorname{Im}\lambda > -A - \delta\log(1+|\lambda|)$ we can further estimate as

$$\leq C\|V\|_{L^\infty} e^{(A+\delta\log(1+|\lambda|))(b-a)}/|\lambda|$$

$$\leq C'\|V\|_{L^\infty}|\lambda|^{-1+\delta(b-a)} \leq 1/2 \text{ if } \delta < \frac{1}{b-a} \text{ and } |\lambda| \geq R.$$

Plugging into 2.3.7 we obtain 2.3.5 if we bound the terms $\rho R_0(\lambda)\rho_1$ and $\rho R_0(\lambda)(1-\rho_1)\rho$ similarly.

The proof of Thm 2.9 will be omitted. The idea is contour deformation, spectral theorem, etc.

2.4 Scattering matrix in dimension one

Let's suppose V is a compactly supported real-valued potential, $\operatorname{supp} V \subseteq [-R, R]$.

Outside the support of V, a sol'n of $(P_V - \lambda^2)u = 0$ can be written as a sum of outgoing and incoming terms $u(x) = u_{in}(x) + u_{out}(x)$, $|x| \geq R$.

Since $V \equiv 0$ in this region, the sol'n is simple, but we adhere to convention on p. 33.

$$u_{in}(x) = b_{\operatorname{sgn}(x)} e^{-i\lambda|x|}, \quad u_{out}(x) = a_{\operatorname{sgn}(x)} e^{i\lambda|x|} \quad |x| \geq R.$$

We want to compare the incoming and outgoing waves, so will consider the scattering matrix $S : \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \longmapsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$

Actually, $S = S(\lambda)$, depends on the frequency λ, and we want to find sol'ns of $P_V u = \lambda^2 u$ everywhere of the form

$u^{\pm}(x) = e^{\pm i\lambda x} + v^{\pm}(x, \lambda)$, where $v^{\pm}(x, \lambda)$ is outgoing.

Now, formally inverting our operator, we see that $v^{\pm}(x, \lambda) = -R_V(\lambda)(Ve^{\pm i\lambda x})$

at least away from the poles — literally just apply the def'n of $R_V(\lambda)$ given in Thm 2.2, as the inverse of $P_V - \lambda^2$.

<u>Remark</u> The $\pm$ notation is because $S^0 = \{1, -1\}$ "=" $\{+, -\}$. In higher dimensions we will replace $\pm$ by $\omega \in S^{n-1}$.

Defining $v^{\pm}_{\operatorname{sgn}(x)}(\lambda) := e^{-i\lambda|x|} v^{\pm}(x, \lambda)$, $|x| > R$

└ to cancel the phase in u_{out}

$$S(\lambda) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 + v^+_+(\lambda) \\ v^+_-(\lambda) \end{pmatrix} \qquad S(\lambda) : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} v^-_+(\lambda) \\ 1 + v^-_-(\lambda) \end{pmatrix}$$

i.e. $S(\lambda) = I + A(\lambda) \quad A(\lambda) = \begin{pmatrix} v_+^+(\lambda) & v_+^- \\ v_-^+(\lambda) & v_-^-(\lambda) \end{pmatrix}$

Thm 2.11 (Scattering matrix in terms of the resolvent)

1) The coefficients of $A(\lambda)$ are meromorphic functions of λ given by the following formulae:

(2.4.8) $v_\theta^\omega(\lambda) = \frac{1}{2i\lambda} \int_{\mathbb{R}} e^{-i\lambda(\omega-\theta)x} V(x)\left(1 - e^{-i\lambda\omega x} R_V(\lambda)(e^{i\lambda\omega \cdot} V)(x)\right) dx.$

where $\theta, \omega \in \{+,-\}$.

2) If we put $E_\rho(\lambda) : L^2(\mathbb{R}) \to \mathbb{C}^2 \quad E_\rho(\lambda)u = \left(\int_{\mathbb{R}} e^{-i\lambda x} u(x)\rho(x)\,dx, \int_{\mathbb{R}} e^{i\lambda x} u(x)\rho(x)\,dx\right)$

where $\rho \in L^\infty_{comp}$, $\rho V = V$ then

$$S(\lambda) = I + \frac{1}{2i\lambda} E_\rho(\lambda)(I + VR_0(\lambda)\rho)^{-1} V E_\rho(\bar{\lambda})^*.$$

Pf: Since $R_V(\lambda) = R_0(\lambda)(I - VR_V(\lambda))$, we have

$v_\theta^\omega(\lambda) = -e^{-i\lambda\theta y} R_0(\lambda)(I - VR_V(\lambda))(Ve^{i\omega\lambda \cdot})(y), \quad \theta y > R$, supp $V \subset [-R,R]$

But we know $R_0(\lambda)f(y) = \frac{-1}{2i\lambda} e^{i\theta\lambda y} \int_{\mathbb{R}} e^{-i\theta\lambda x} f(x)\,dx \quad \theta y > R$

and combining these gives 2.48.

To get (2) use $R_V(\lambda)V = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}V \quad (2.2.9)$

and $(I + VR_0(\lambda)\rho)^{-1}V = \rho(I + VR_0(\lambda)\rho)^{-1}V$ and plug in to

get $v_\theta^\omega(\lambda) = -e^{-i\lambda\theta y} R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}(Ve^{i\omega\lambda \cdot})$

Remarks: The coefficients $\sigma_\theta^\omega(\lambda)$ have important physical interpretations: 3

$t(\lambda) = 1 + \sigma_+^+(\lambda)$ is the transmission coefficient

$r_+(\lambda) = \sigma_+^-(\lambda)$ is the right reflection coefficient

$r_-(\lambda) = \sigma_-^+(\lambda)$ is the left reflection coefficient.

Changing λ to $-\lambda$ in the def'n of $S(\lambda)$ shows that

(2.4.13) $\quad S(-\lambda) = J S(\lambda)^{-1} J \quad J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ when $S(\lambda), S(-\lambda)$ exist.

When V is real and $\lambda \in \mathbb{R} \setminus \{0\}$ it can be shown that $S(\lambda)^* = S(\lambda)^{-1}$, i.e., $S(\lambda)$ is unitary. Meromorphic continuation then gives

(2.4.14) $\quad V \in L^\infty_{comp}(\mathbb{R};\mathbb{R}) \Rightarrow S(\bar{\lambda})^* = S(\lambda)^{-1} \quad \lambda \in \mathbb{C}$

which implies that $\sigma_\theta^\omega(\lambda)$ are holomorphic for $\lambda \in \mathbb{R}$

$\pm$ should be thought of as elements of the sphere $\mathbb{S}^0$. In higher dimensions similar formulas are valid but now $\theta, \omega \in \mathbb{S}^{n-1}$. The scattering "matrix" is then given as the sum of the identity and a certain integral operator with kernel in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

Using Thm 2.11 we get the following estimates for the coefficients of the scattering matrix:

Thm 2.12 (Estimates on the scattering matrix): For $\operatorname{Im}\lambda \geq 0$ and $|\lambda| \geq C_0$ we have $\|e^{\mp i\lambda x} R_V(\lambda) V e^{\pm i\lambda \theta}\|_{L^2 \to L^2} \leq \frac{C_1}{|\lambda|}$

Combining with (2.4.8, the formula for σ_θ^ω) implies:

$\sigma_+^+(\lambda) = \frac{1}{2i\lambda}\left(\hat{V}(0) + \mathcal{O}\left(\frac{1}{|\lambda|}\right)\right)$

pole or zero of $\det S(\lambda)$ is related to the multiplicity of a scattering resonance

Pf The kernel of the integral operator $R_0^\omega(\lambda) := e^{-i\lambda\omega x} R_0(\lambda) e^{i\lambda\omega\cdot}$ is given by $R_0^\omega(\lambda, x, y) := e^{-i\lambda\omega x} R_0(\lambda, x, y) e^{i\lambda\omega y} = \frac{i}{2\lambda} e^{i\lambda(|x-y| - \omega(x-y))}$

As $|x-y| - \omega(x-y) \geq 0$, $\|VR_0^\omega(\lambda)e\|_{L^2 \to L^2} \leq \frac{C}{|\lambda|}$, for $\operatorname{Im}\lambda \geq 0$, implies that the Neumann series for $(I + VR_0^\omega(\lambda)e)^{-1}$ converges when $\operatorname{Im}\lambda \geq 0$ and $|\lambda|$ is sufficiently big, say, $|\lambda| > C_0$.

Similarly $R_0^\omega(\lambda)e = \mathcal{O}(1/|\lambda|)$ as an operator $L^2 \to L^\infty$, when $\operatorname{Im}\lambda \geq 0$.

Recalling (2.2.9) which says that $R_V(\lambda)V = R_0(\lambda)(I + VR_0(\lambda)e)^{-1}V$

We see that
$$e^{-i\omega\lambda x} R_V(\lambda) V e^{i\omega\lambda\cdot} = e^{-i\omega\lambda x} R_0(\lambda) e^{i\omega\lambda\cdot} (e^{-i\omega\lambda\cdot}(I + VR_0(\lambda))e)^{-1} e^{i\omega\lambda}$$
$$= R_0^\omega(\lambda)(I + VR_0^\omega(\lambda)e)^{-1} V,$$

when $\operatorname{Im}\lambda > 0$ and $|\lambda| > C_0$. Now take norm.

One can also relate the determinant of the scattering matrix to the determinant $D(\lambda) := \det(I + VR_0(\lambda)e)$ from Yonah's section.

<u>Thm 2.13 (A determinant identity)</u> For $V, e \in L^\infty_{comp}$ satisfying $eV = V$ let $D(\lambda) := \det(I + VR_0(\lambda)e)$. Then $\frac{D(-\lambda)}{D(\lambda)} = \det S(\lambda)$.

The proof is direct computation + the fact that $\det(I + AB) = \det(I + BA)$.

When combined with some stuff in Yonah's section it has the following corollary:

Thm (Multiplicity of scattering poles [poles of the scattering matrix] in one dimension:

The multiplicity of a scattering pole defined by $m_S(\lambda) = \frac{-1}{2\pi i} \operatorname{tr} \oint_\lambda S(\zeta)^{-1} \partial_\zeta S(\zeta) d\zeta$ where the integral is over a positively oriented circle which includes λ and no other pole or zero of $\det S(\lambda)$ is related to the multiplicity of a scattering resonance

$\hookrightarrow m_S(\lambda) = m_R(\lambda) = -m_R(-\lambda)$

$\llcorner$ defined in Yonah's talk $m_R(\lambda) := \operatorname{rank} \oint_{\lambda} R_V(\zeta)\, d\zeta$

One can also show that $\det S(0) = (-1)^{m_R(0)+1}$, i.e., compute the zero resonance in terms of the scattering matrix.